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Journal of Geometry and Physics 57 (2007) 597–615
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Belorousski–Pandharipande relation in dGBV algebras

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Received 8 September 2005; received in revised form 26 February 2006; accepted 5 May 2006
Available online 30 June 2006

Abstract

We prove that the genus expansion of solutions of the WDVV equation constructed from dGBV algebras satisfies the differential equation determined by the Belorousski–Pandharipande relation in cohomology of the moduli space of curves $\overline{\mathcal{M}}_{2,3}$.

MSC: 53D45

Keywords: WDVV equation; Genus expansion; dGBV algebras

1. Introduction

In papers [3,4], the algebraic formalization of Zwiebach invariants gives a purely algebraic construction of the genus expansion of solutions of the WDVV equation in terms of cH-algebras. Since we know that Zwiebach invariants induce Gromov–Witten invariants on subbicomplexes with zero differential, it is naturally of concern to check different relations coming from the geometry of the moduli space of curves.

In Gromov–Witten theory, we represent a solution of the WDVV equation as a generating function for genus 0 Gromov–Witten invariants of a suitable algebraic variety (without ψ-classes). Then we consider Gromov–Witten invariants with ψ-classes and in arbitrary genus. In this approach, any relation among natural strata in cohomology of the moduli space of curves gives us a differential equation for the Gromov–Witten potential. This property is just a corollary of the splitting axiom.

In our construction, we do not have a definition that can be compared with the full Gromov–Witten potential. We just define genus expansion with descendants (ψ-classes) only at one point. But some relations from Gromov–Witten theory are already nontrivial even for this reduced genus expansion. In particular, it is enough to have descendants at one point to pose a question on Belorousski–Pandharipande relations among codimension 2 strata in $\overline{\mathcal{M}}_{2,3}$ [1].

In this paper, we prove that our genus expansion satisfies the differential equation defined by the Belorousski–Pandharipande relation. In fact, it is the unique currently known relation in genera ≤ 2 that makes sense in our construction and that we have not yet checked in [3,4].

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0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.05.002

Since this paper is just a sequel to [3,4], we refer the reader to those papers for the origin, motivation, and more detailed exposition of the new construction that we study here. Also we mention that in this paper we retain all earlier mathematical problems of this theory such as the lack of examples and the problem of convergence.

This paper is organized as follows. In Section 2, we recall our construction of genus expansion (in fact, only its parts used in this paper). In Section 3, we recall the Belorousski–Pandharipande relation and formulate our main theorem. In the rest of the paper, we prove (or rather outline the proof of) our theorem.

2. Construction of the potential

In our case, the Belorousski–Pandharipande relation is a differential equation for four different formal power series: generating functions for the correlators in genera 0, 1, and 2 without descendants (Φ_0, Φ_1, and Φ_2, respectively), and the generating function for the correlators in genus 2 with one descendant at one point ($\Phi_2^{(1)}$). Our goal in this section is to define these four formal power series.

In Section 2.1 we explain what a cH-algebra is and fix notation for all necessary operators in it. In Section 2.2 we explain how we use graphs to encode tensor expressions. In Section 2.3 we fix notation for all tensors in cH-algebras that we use in this paper. Also we discuss there a subtlety related to the signs. Then in Section 2.4 we define Φ_0, Φ_1, Φ_2, and $\Phi_2^{(1)}$, and give precise formulas (in terms of graphs) for the first few terms of these power series.

In fact, it can also be useful to study the definition of the full potential with descendants only at one point given in [4]. But the full definition given there is rather involved and appears to be natural only in the course of the calculations in [4] or in the framework of Zwiebach invariants [3].

2.1. cH-algebras

In this section, we recall the definition of cH-algebras [3,4]. A supercommutative associative $\mathbb{C}$-algebra H is called a cH-algebra if there are two odd linear operators $Q, G_-: H \to H$ and an integral $\int: H \to \mathbb{C}$ satisfying the following axioms:

(1) $Q^2 = G_-^2 = QG_- + G_-Q = 0$;
(2) $H = H_0 \oplus H_4$, where $QH_0 = G_-H_0 = 0$ and H_4 is represented as a direct sum of subspaces of dimension 4 generated by $e_\alpha, Qe_\alpha, G_-e_\alpha, QG_-e_\alpha$ for some vectors $e \in H_4$, i.e. $H_4 = \bigoplus_\alpha \langle e_\alpha, Qe_\alpha, G_-e_\alpha, QG_-e_\alpha\rangle$ (Hodge decomposition);
(3) Q is an operator of the first order, it satisfies the Leibniz rule: $Q(ab) = Q(a)b + (-1)^{\tilde{a}} aQ(b)$ (here and below we denote by $\tilde{a}$ the parity of $a \in H$);
(4) G_- is an operator of the second order, it satisfies the seven-term relation: $G_-(abc) = G_-(ab)c + (-1)^{\tilde{b}(\tilde{a}+1)} bG_-(ac) + (-1)^{\tilde{a}} aG_-(bc) - G_-(a)bc - (-1)^{\tilde{a}} aG_-(b)c - (-1)^{\tilde{a}+\tilde{b}} abG_-(c)$;
(5) G_- satisfies the property called the 1/12-axiom: $\mathrm{str}(G_- \circ a\cdot) = (1/12)\mathrm{str}(G_-(a)\cdot)$ (here $a\cdot$ and $G_-(a)\cdot$ are the operators of multiplication by a and $G_-(a)$ respectively).

We define an operator $G_+: H \to H$. We put $G_+H_0 = 0$; on each subspace $\langle e_\alpha, Qe_\alpha, G_-e_\alpha, QG_-e_\alpha\rangle$, we define G_+ as $G_+e_\alpha = G_+G_-e_\alpha = 0$, $G_+Qe_\alpha = e_\alpha$, and $G_+QG_-e_\alpha = G_-e_\alpha$. We see that $[G_-, G_+] = 0$; $\Pi_4 = [Q, G_+]$ is the projection to H_4 along H_0; $\Pi_0 = \mathrm{Id} - \Pi_4$ is the projection to H_0 along H_4.

An integral on H is an even linear function $\int: H \to \mathbb{C}$. We require $\int Q(a)b = (-1)^{\tilde{a}+1} \int aQ(b)$, $\int G_-(a)b = (-1)^{\tilde{a}} \int aG_-(b)$, and $\int G_+(a)b = (-1)^{\tilde{a}} \int aG_+(b)$. These properties imply that $\int G_-G_+(a)b = \int aG_-G_+(b)$, $\int \Pi_4(a)b = \int a\Pi_4(b)$, and $\int \Pi_0(a)b = \int a\Pi_0(b)$.

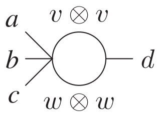
We can define a scalar product on H: $(a, b) = \int ab$. We suppose that this scalar product is non-degenerate. Using the scalar product we may turn an operator $A : H \to H$ into a bivector that we denote by $[A]$.

2.2. Tensor expressions in terms of graphs

Here we explain a way to encode some tensor expressions over an arbitrary vector space in terms of graphs.

Consider an arbitrary graph (we allow graphs to have leaves and we require vertices to be of degree at least 3). We associate a symmetric n-form with each internal vertex of degree n, a symmetric bivector with each edge, and a vector with each leaf. Then we can substitute the tensor product of all vectors in leaves and bivectors in edges into the product of n-forms in vertices, distributing the components of tensors in the same way as the corresponding edges and leaves are attached to vertices in the graph. This way we get a number.

Let us study an example:



(1)

We assign a 5-form x to the left vertex of this graph and a 3-form y to the right vertex. Then the number that we get from this graph is $x(a, b, c, v, w) \cdot y(v, w, d)$.

Note that the vectors, bivectors and n-forms used in this construction can depend on some variables. Then what we get is not a number, but a function.

2.3. Usage of graphs in cH-algebras

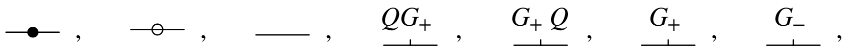
Consider a cH-algebra H. There are some standard tensors over H, which we associate with elements of graphs below. Here we introduce the notation for these tensors.

We always assign the form

$$(a_1, \dots, a_n) \mapsto \int a_1 \cdot \dots \cdot a_n \tag{2}$$

to a vertex of degree n.

There are a collection of bivectors that will arise below at edges: $[G_-G_+]$, $[\Pi_0]$, $[Id]$, $[QG_+]$, $[G_+Q]$, $[G_+]$, and $[G_-]$. In pictures, edges with these bivectors will be denoted by



(3)

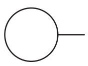
respectively. Note that an empty edge corresponding to the bivector $[Id]$ can usually be contracted (if it is not a loop).

The vectors that we will put at leaves depend on some variables. Let $\{e_1, \dots, e_s\}$ be a homogeneous basis of H_0. With each vector e_i we associate two formal variables, $T_{0,i}$ and $T_{1,i}$, of the same parity as e_i. Then we will put at a leaf either the vector $E_0 = \sum e_i T_{0,i}$ (denoted by an empty leaf) or the vector $E_1 = \sum e_i T_{1,i}$ (denoted by an arrow at the leaf).

2.3.1. Remark

There is a subtlety related to the fact that H is a $\mathbb{Z}_2$-graded space. In order to give an honest definition we must do the following. Suppose we consider a graph of genus g. We can choose g edges in such a way that the graph being cut at these edge turns into a tree. To each of these edges we have already assigned a bivector $[A]$ for some operator $A\colon H \to H$. Now we have to put the bivector $[JA]$ instead of the bivector $[A]$, where J is an operator defined by the formula $J\colon a \mapsto (-1)^{\tilde{a}} a$.

In particular, consider the following graph (this is also an example of the notation given above):



(4)

An empty loop corresponds to the bivector $[Id]$. An empty leaf corresponds to the vector E_0. A trivalent vertex corresponds to the 3-form given by the formula $(a, b, c) \mapsto \int abc$.

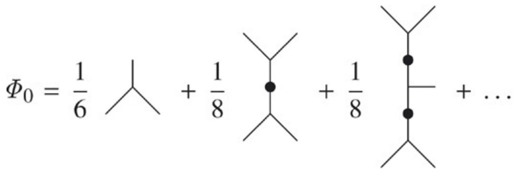
If we ignore this remark, then what we get is just the trace of the operator $a \mapsto E_0 \cdot a$. But using this remark we get the supertrace of this operator.

In fact, this subtlety will play no role in this paper. It affects only some signs in calculations and all these signs will be hidden in lemmas shared from [3,4]. So, one can just ignore this remark.

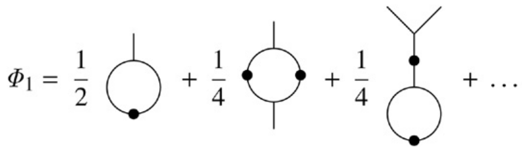
2.4. Construction of Φ_0, Φ_1, Φ_2, and $\Phi_2^{(1)}$

Now we describe Φ_0, Φ_1, Φ_2, and $\Phi_2^{(1)}$ using the notation given above.

The formal power series Φ_0 (Φ_1, Φ_2) is just the sum over all trivalent graphs of genus 0 (1, 2, respectively) with empty leaves and edges with thick black dots. By each graph we put a coefficient equal to the inverse of the number of its automorphisms.



(5)



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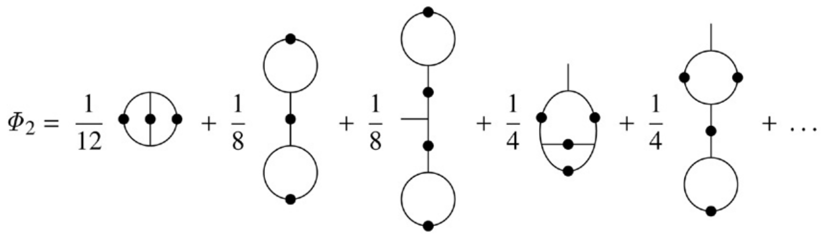
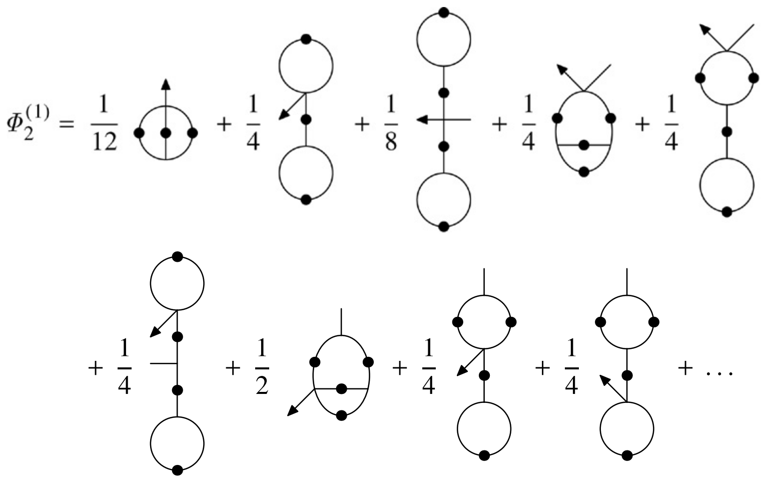


(7)

The formal power series $\Phi_2^{(1)}$ is the sum over graphs of genus 2 with edges with thick black dots satisfying some additional conditions. First, there is exactly one vertex of degree 4 and all of the vertices are trivalent. Second, at this vertex of degree 4, there is a leaf with an arrow. Third, all other leaves are empty. Each graph is weighted with the inverse of the number of its automorphisms fixing the leaf with the arrow.



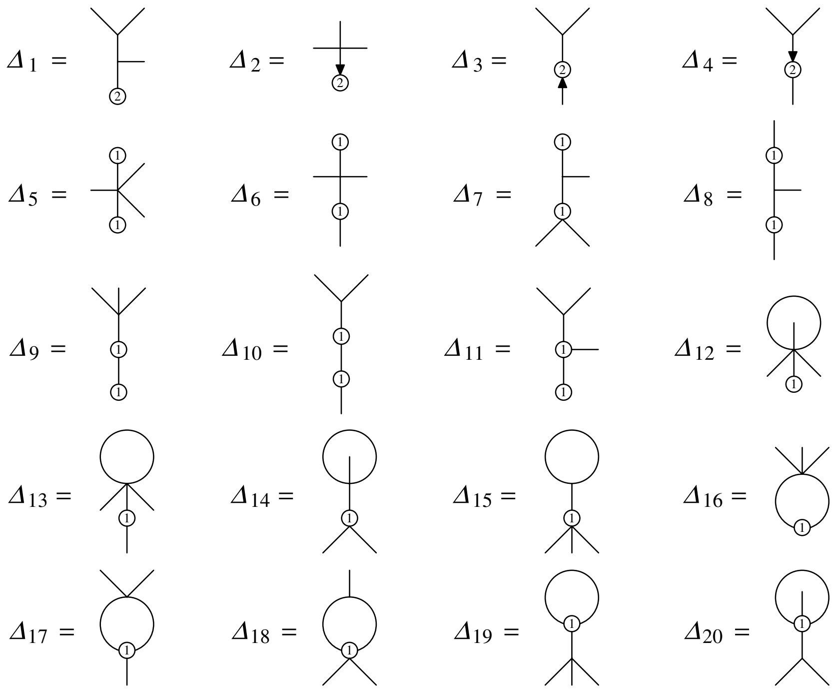
(8)

In fact, in order to obtain an expression for $\Phi_2^{(1)}$ one can just take the expression for Φ_2 and add an additional leaf with an arrow in all possible ways to each graph. Also it is obvious that $\Phi_2^{(1)}$ is linear in $T_{1,i}$, $i = 1, \ldots, s$.

3. Belorousski–Pandharipande relation

3.1. Notation

The Belorousski–Pandharipande relation is a relation in (co)homology of $\overline{\mathcal{M}}_{2,3}$ between the cycles of natural strata of complex codimension 2 in $\overline{\mathcal{M}}_{2,3}$. Below, we list the strata participating in the Belorousski–Pandharipande relation:



We explain our notation. Note that the graphs here have completely different meaning to all other graphs in this paper. We use the language of dual graphs, that is, vertices correspond to irreducible curves, edges correspond to points of intersection, leaves correspond to marked points. A thick vertex labeled 2 corresponds to a genus 2 curve; a thick vertex labeled 1 corresponds to a genus 1 curve. A simple vertex corresponds to a genus 0 curve. An arrow on an edge or a leaf means that we take the ψ-class at the destination of the arrow.

This way to describe strata in the moduli space of curves was introduced by E. Getzler, see [2].

For example, consider the picture of Δ_1. A generic point of this stratum is represented by a three-component curve such that one component has genus 0; there is one marked point on this curve and two other curves intersect it. One of the other curves has genus 0 and two marked points; another curve has genus 2 and no marked points.

Another example. We consider the picture of Δ_2. A generic point of this stratum is represented by a two-component curve; one component has genus 0; there are three marked points and one point of intersection with another curve. Another curve has genus 2; there are no marked points, but we take the ψ-class on this curve at the point of intersection.

One more example is the picture of Δ_3. A generic point of this stratum is represented by a two-component curve; one curve has genus 0, two marked points, and one point of intersection with another curve. Another curve has genus 2, one point of intersection with the first curve, and one marked point with the ψ-class.

3.2. The relation

We recall the Belorousski–Pandharipande relation:

$$\begin{aligned}&-4\Delta_1+12\Delta_2+6\Delta_3-6\Delta_4+\frac{12}{5}\Delta_5-\frac{12}{5}\Delta_6+\frac{24}{5}\Delta_7-\frac{36}{5}\Delta_8-\frac{36}{5}\Delta_9+\frac{18}{5}\Delta_{10}-\frac{12}{5}\Delta_{11}\\&\quad+\frac{1}{10}\Delta_{12}-\frac{3}{10}\Delta_{13}+\frac{3}{10}\Delta_{14}-\frac{1}{10}\Delta_{15}+\frac{6}{5}\Delta_{16}-\frac{6}{5}\Delta_{17}+\frac{2}{5}\Delta_{18}-\frac{3}{5}\Delta_{19}-\frac{1}{5}\Delta_{20}=0.\end{aligned}\tag{9}$$

One can note that the coefficients in Eq. (9) do not coincide with the coefficients of the initial relation in [1]. This is for two reasons. First, we do not weight the strata in the formula with the inverse order of the automorphism group of their generic point. Second, we consider each possible enumeration of marked points only once, without multiplicities. We refer the reader to [1] for the explanation of the conventions that we do not keep here.

3.3. Differential equation

As we have already explained in the introduction, the Belorousski–Pandharipande relation gives us a differential equation for Φ_0, Φ_1, Φ_2, and $\Phi_2^{(1)}$. We illustrate this correspondence with examples.

If all variables are even, we have:

$$\Delta_1 \rightsquigarrow \frac{\partial \Phi_2}{\partial T_{0,i}} \eta_{ij} \frac{\partial^3 \Phi_0}{\partial T_{0,j} \partial T_{0,a} \partial T_{0,k}} \eta_{kl} \frac{\partial^3 \Phi_0}{\partial T_{0,l} \partial T_{0,b} \partial T_{0,c}} + 2 \text{ terms obtained by permutations of } \{a, b, c\}, \tag{10}$$

$$\Delta_2 \rightsquigarrow \left(\frac{\partial \Phi_2^{(1)}}{\partial T_{1,i}} - \frac{\partial^2 \Phi_0}{\partial T_{0,i} \partial T_{0,k}} \eta_{kl} \frac{\partial \Phi_2}{\partial T_{0,l}} \right) \eta_{ij} \frac{\partial^4 \Phi_0}{\partial T_{0,j} \partial T_{0,a} \partial T_{0,b} \partial T_{0,c}} \tag{11}$$

$$\Delta_3 \rightsquigarrow \left(\frac{\partial^2 \Phi_2^{(1)}}{\partial T_{1,a} \partial T_{0,i}} - \frac{\partial^2 \Phi_0}{\partial T_{0,a} \partial T_{0,k}} \eta_{kl} \frac{\partial^2 \Phi_2}{\partial T_{0,l} \partial T_{0,i}} \right) \eta_{ij} \frac{\partial^3 \Phi_0}{\partial T_{0,j} \partial T_{0,b} \partial T_{0,c}} + 2 \text{ terms obtained by permutations of } \{a, b, c\}, \tag{12}$$

and so on. The metric η_{ij} used here is given by the scalar product on H_0. We have: $\eta_{ij} = (e_i, e_j)$, $\eta^{ij} = [\Pi_0]$.

Note that Δ_2 is defined with the help of one ψ-class on $\overline{\mathcal{M}}_{2,1}$. Let $\pi: \overline{\mathcal{M}}_{2,n} \to \overline{\mathcal{M}}_{2,1}$ be the projection forgetting all marked points except for the first one. Then there is a formula relating ψ_1 and $\pi^*\psi_1$ on $\overline{\mathcal{M}}_{2,n}$. So, the differential expressions corresponding to these strata rely on this formula, which is exactly the first factor in Expression (11). The same remark concerns Δ_3, Δ_4, and the pull-back of ψ_1 from $\overline{\mathcal{M}}_{2,2}$. In this case, the required formula is the first factor of Expression (12).

3.4. Theorem

We state our theorem.

Theorem 1. *Φ_0, Φ_1, Φ_2, and $\Phi_2^{(1)}$ satisfy the Belorousski–Pandharipande relation.*

4. Proof

In this section we prove Theorem 1. The proof is organized in two steps. First we consider the differential equation determined by the Belorousski–Pandharipande relation at the zero point. It is proved by a straightforward calculation with tensors.

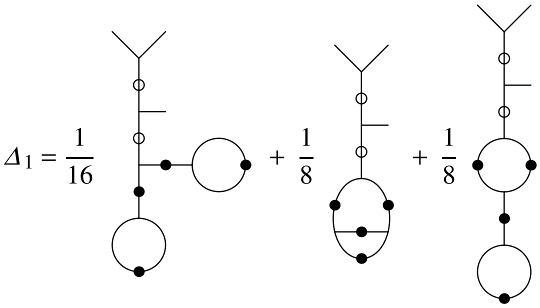
Then we can use the universal technique developed in [3]. That is, for any differential equation proved at the zero point by the same type of calculation as given below, we immediately obtain its proof at any point. This was done very carefully for the WDVV equation and less carefully for the Getzler relation in the last section of [3], and the argument for the Belorousski–Pandharipande relation is literally the same.

So, in Sections 4.1 and 4.4 we discuss only the simplest case of the Belorousski–Pandharipande relation. In Section 4.1 we rewrite it in terms of graphs; in Section 4.2 we explain what kind of calculation is to be done; in Sections 4.3 and 4.4 we give an example of such a calculation for one stratum and discuss it for the other strata.

Finally, in Section 4.5 we recall the basic idea from the last section of [3] that completes the proof of any relation of this type.

4.1. *Relation in terms of graphs*

Consider the degree 0 term of the power series obtained from Φ_0, Φ_1, Φ_2, and $\Phi_2^{(1)}$ using the differential operator determined by Δ_i. Slightly abusing the notation, we denote it also by Δ_i. Then we have:



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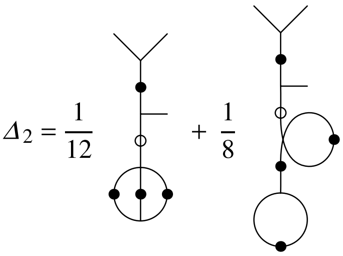

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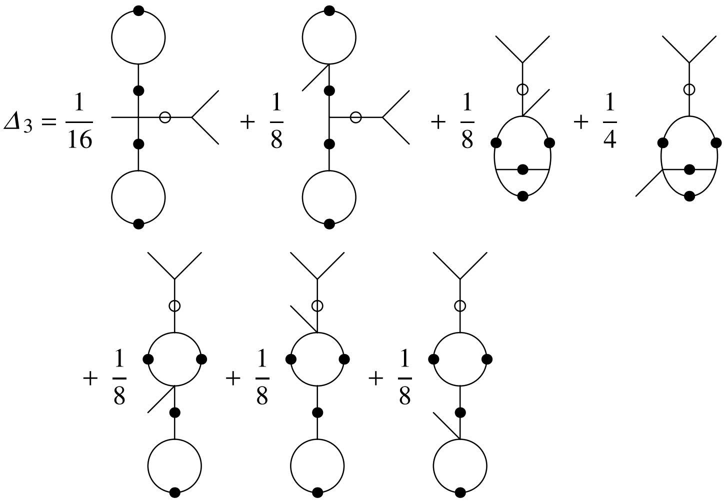


(15)

and so on. We recall that a thick white point on an edge denotes the bivector $[\Pi_0]$ (or $[J\Pi_0]$).

4.2. *Outline of the calculations*

We explain the proof of the simplest case of Theorem 1. We have already expressed each Δ_i at the zero point in terms of graphs with one or two edges marked by $[\Pi_0]$. Using the Leibniz rule for Q and the seven-term relation and $1/12$-axiom for G_-, we get out of $[\Pi_0]$ in our expressions. In this way we obtain an expression for each Δ_i in terms of 60 *final graphs*. Then we substitute these expressions in Belorousski–Pandharipande relation (9), and we see that the coefficient of each final graph in this relation is equal to 0. This proves the simplest case of our theorem.

The final graphs are listed in Appendix A; final expressions for Δ_i are given in Appendix B. Below, we explain how to get out of $[\Pi_0]$ in our graphs by way of an example (we give detailed calculations for Δ_3).

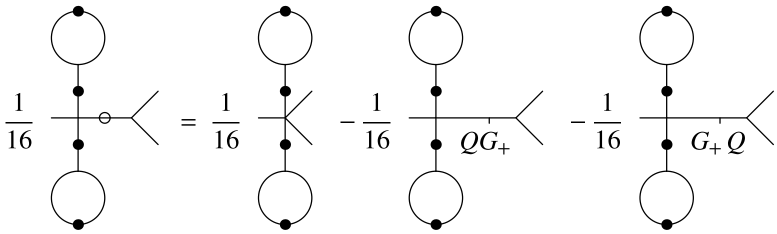
4.3. *Calculations for Δ_3*

We consider the right hand side of Eq. (15). Our goal is to get out of thick white points in these graphs. Finally, we must obtain an expression in terms of graphs from Appendix A.

We carry out our calculations in two steps. At the first step we consider each graph of the right hand side of Eq. (15) separately. At the second step we arrange the results of the first step and obtain an expression in final graphs.

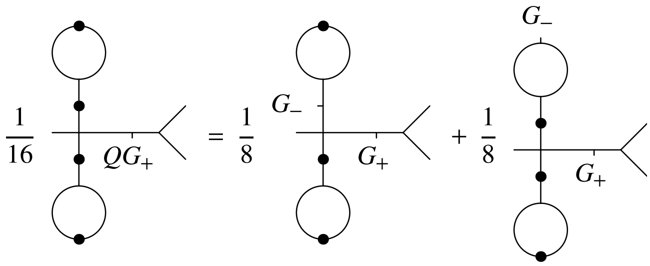
4.3.1. *First step for the first picture*

We recall that $\Pi_0 = \mathrm{Id} - QG_+ - G_+Q$. Also we note that if we have an edge (not a loop) marked by [Id], then we can contract this edge. We have:



(16)

We recall that $[Q, G_-G_+] = -G_-$ and $Qe_i = 0$ for any i. Using these properties, the Leibniz rule for Q, and taking into account the symmetries of our graphs, we have:



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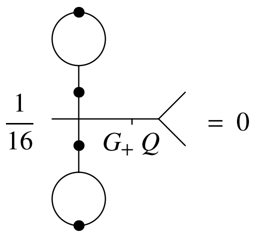
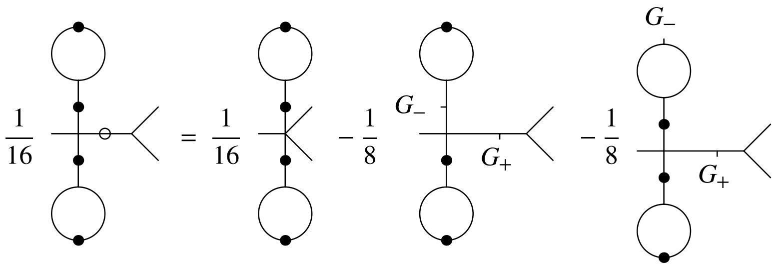


(18)

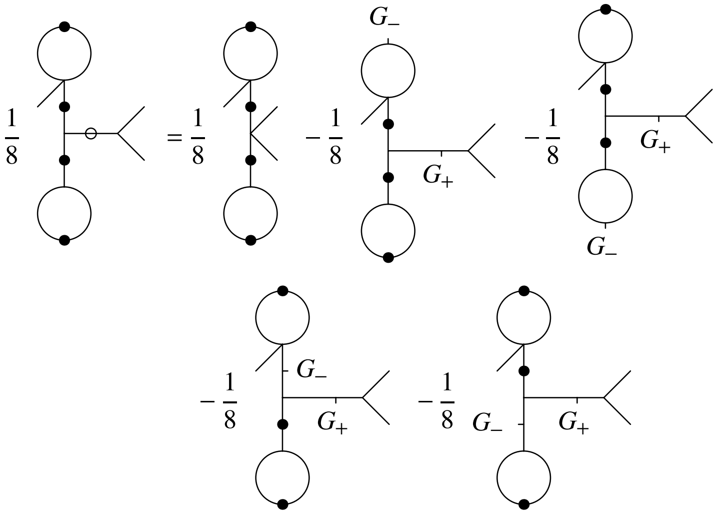
Therefore,



(19)

4.3.2. First step for all other pictures

The same calculations for all other pictures give us:



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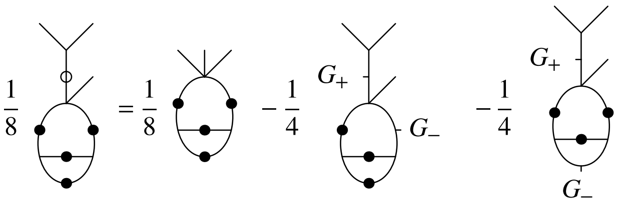

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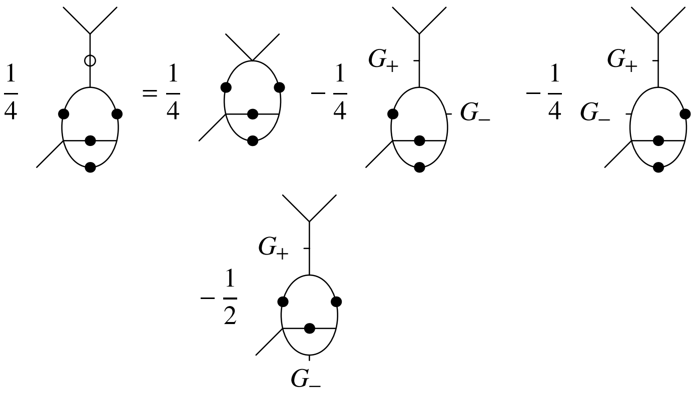

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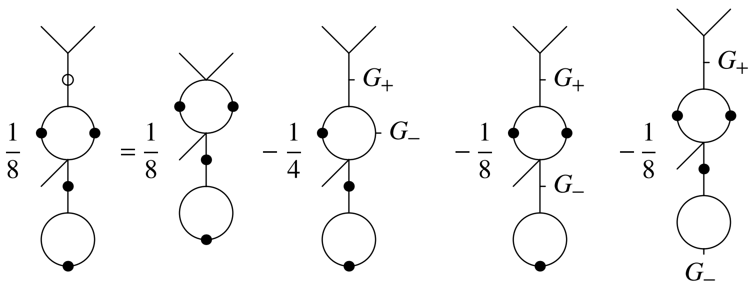

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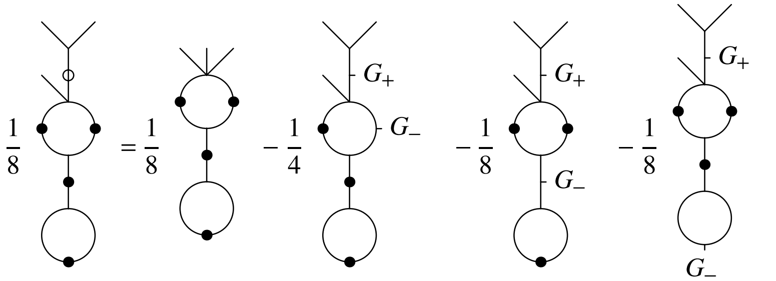

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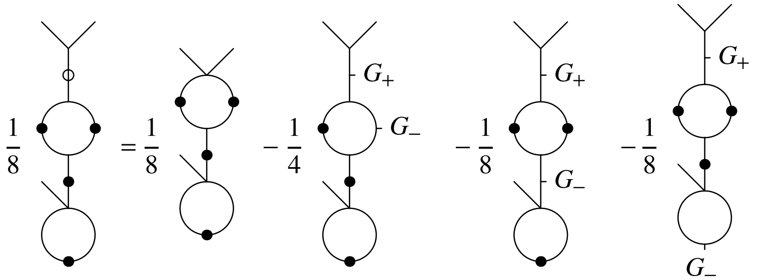
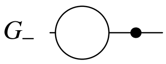


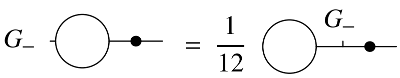
(25)

4.3.3. Corollaries of the 1/12*-axiom*

In this section, we prove that some graphs in Eqs. (19)–(25) are equal to 0.

Lemma 1. *Vector*  *is equal to* 0.

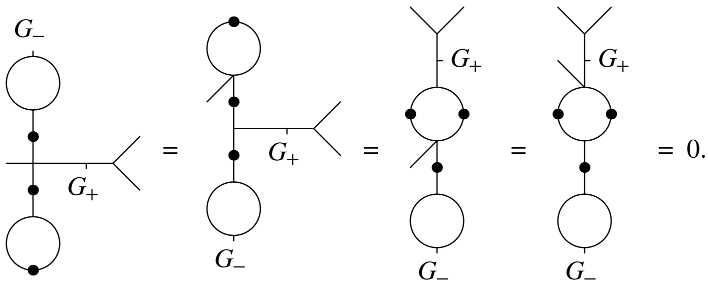
Proof. Indeed, from 1/12-axiom, it follows that



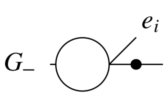
(26)

Since $G_-G_-G_+ = 0$, the last vector is equal to zero. □

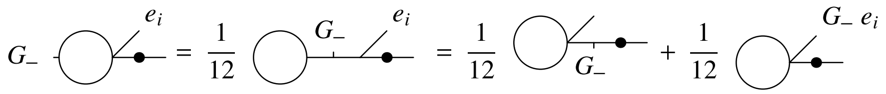
From this lemma, it is obvious that



(27)

Lemma 2. *For any i vector*  *is equal to* 0.

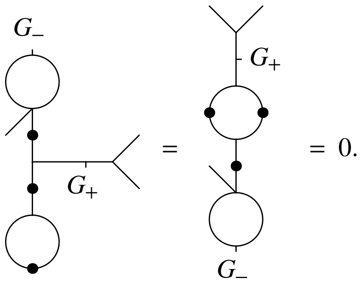
Proof. First, we apply the 1/12-axiom, and then we apply the auxiliary lemma from [4]. We have:



(28)

Since $G_-G_-G_+ = 0$ and $G_-e_i = 0$, the last two vectors are equal to zero. □

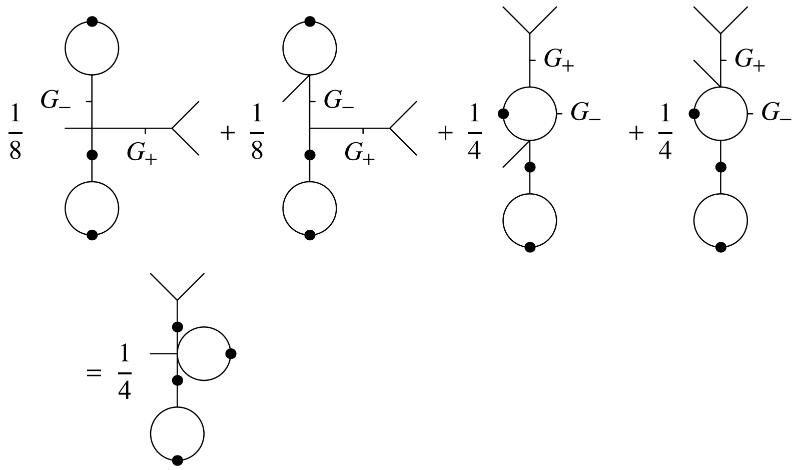
From this lemma, it is obvious that



(29)

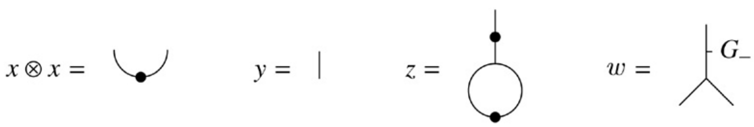
4.3.4. Corollaries of the seven-term relation

In this section, we list some corollaries of the seven-term relation. We have:



(30)

We prove this formula. For convenience, we split all these graphs into the same tensor pieces. We list the notation for these tensor pieces:



(31)

Note that x, y, z are even vectors, but w is an odd one. Eq. (30) is equivalent to

$$\frac{1}{8}\int G_-(x^2)yzw + \frac{1}{8}\int G_-(x^2y)zw + \frac{1}{4}\int G_-(xyz)xw + \frac{1}{4}\int G_-(xz)xyw = \frac{1}{4}\int G_-(x^2yz)w. \quad (32)$$

Also note that $G_-(x) = G_-(y) = G_-(z) = 0$. Then from the seven-term relation, it follows that

$$G_-(x^2yz) = 2G_-(xy)xz + 2G_-(xz)xy + G_-(x^2)yz + G_-(yz)x^2 \quad (33)$$

$$G_-(x^2y) = 2G_-(xy)x + G_-(x^2)y \quad (34)$$

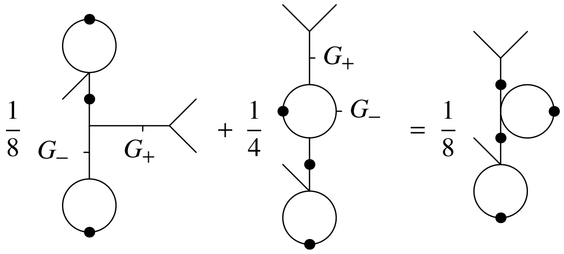
$$G_-(xyz) = G_-(xy)z + G_-(xz)y + G_-(yz)x. \quad (35)$$

Substituting Eqs. (34) and (35) in the left hand side of Eq. (32), we get:

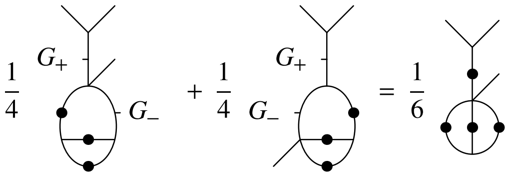
$$\int \left(\frac{1}{4} G_-(x^2) yzw + \frac{1}{2} G_-(xy) xzw + \frac{1}{2} G_-(xz) xyw + \frac{1}{4} G_-(yz) x^2 w \right). \tag{36}$$

Substituting Eq. (33) in the right hand side of Eq. (32), we get exactly the same. This proves Eq. (32) and therefore Eq. (30).

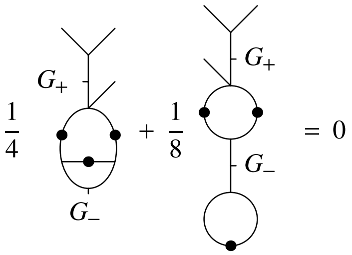
We prove in the same way that



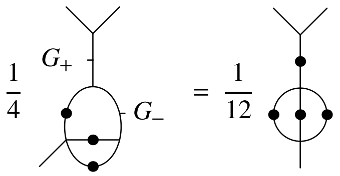
(37)



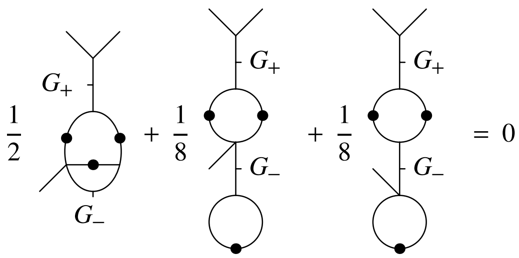
(38)



(39)



(40)



(41)

4.3.5. *Final formula for Δ_3*

Using Eqs. (15)–(41) and the notation for the final graphs from Appendix A, we get:

$$\Delta_3 = \frac{1}{8} A_1 + \frac{1}{16} A_2 + \frac{1}{8} A_3 + \frac{1}{8} F_1 + \frac{1}{8} F_2 + \frac{1}{8} F_3 + \frac{1}{4} F_4 - \frac{1}{4} D_2 - \frac{1}{6} D_3 - \frac{1}{8} H_1 - \frac{1}{12} H_2. \tag{42}$$

4.4. *The other Δ_i*

We carry out the same calculation for all other Δ_i. If there are two thick white points in graphs for Δ_i, then we get out of them successively. The results of these calculations are arranged in tables in Appendix B.

If we substitute all these expressions for Δ_i in the Belorousski–Pandharipande relation, we get zero identically. This proves our theorem.

For a much more detailed exposition of our calculations, see [5].

4.5. *Reconstruction of the full proof*

Now we explain what to do when parameters are not set to zero. In terms of graphs, this means that for any Δ_i we are to consider graphs with the same structure as before but with an arbitrary number of additional leaves.

In [3] the authors notice that these additional leaves can be gathered into some special operators. That is, instead of considering graphs with an arbitrary number of additional leaves, we can consider the same graphs as in the simplest case, but we replace the vectors E_0 and E_1 on leaves and bivectors $[G_-G_+]$ and $[\Pi_0]$ on edges with new complicated vectors and bivectors.

These new vectors and bivectors depend on parameters and can be written down explicitly in terms of the Barannikov–Kontsevich solution of the Maurer–Cartan equation as is done in [3].

Here is one subtlety related to the strata with one ψ-class. At this step we have simultaneously switched from ψ-classes to kinds of pull-backs of ψ-classes. But it was proved in [4] that these pull-backs are related to ψ-classes via exactly the same formulas as in Gromov–Witten theory!

So, we take the same graphs as in the simplest case, we put new vectors on leaves and bivectors on edges, and we must prove *exactly the same* relation as in the simplest case.

The main feature of this approach is that the properties of these new vectors and bivectors are almost the same as the properties of E_0, E_1, $[G_-G_+]$, and $[\Pi_0]$. So we can just repeat our argument for getting out of thick white points in graphs.

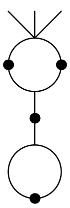
We refer the reader to the last section of [3] for the precise formulas for these new vectors and bivectors and lemmas describing their properties. In fact, this reconstruction of the full proof works for a rather large class of differential equations in cH-algebras; in particular all possible relations coming from the geometry of the moduli space of curves are definitely in this class.

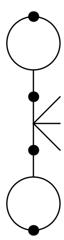
Acknowledgements

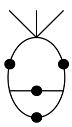
S.S. was partially supported by the grants RFBR-05-01-02806-CNRS-a, RFBR-04-02-17227-a, NSh-4719.2006.1, NWO-RFBR-047.011.2004.026 (RFBR-05-02-89000-NWO-a), MK-5396.2006.1, by the Göran Gustafsson foundation, and by P. Deligne's fund based on his 2004 Balzan prize. I.S. was partially supported by grant RFFR-06-01-00037-a.

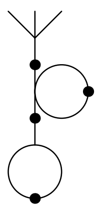
We are very grateful to A.S. Losev for numerous helpful remarks and discussions.

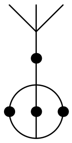
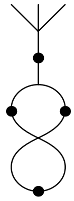
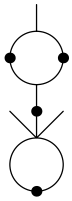
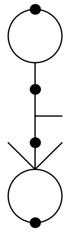
Appendix A. Final graphs

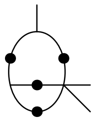
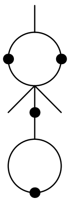
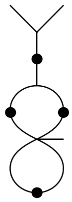
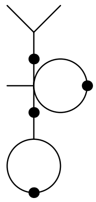
$A_1 =$ 

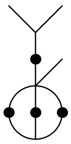
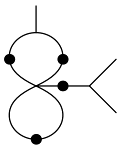
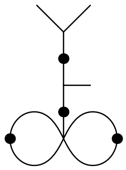
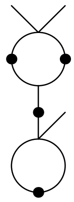
$A_2 =$ 

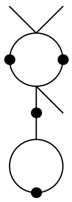
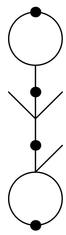
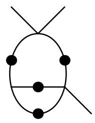
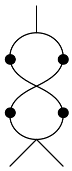
$A_3 =$ 

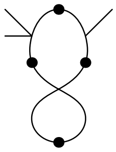
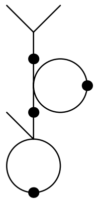
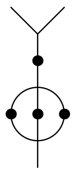
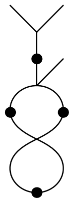
$B_1 =$ 

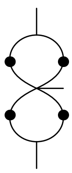
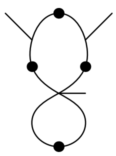
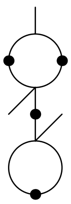
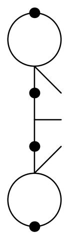
$B_2 =$ 
$B_3 =$ 
$C_1 =$ 
$C_2 =$ 

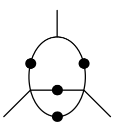
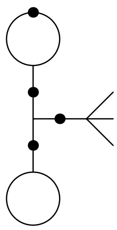
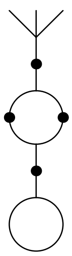
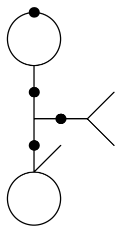
$C_3 =$ 
$C_4 =$ 
$D_1 =$ 
$D_2 =$ 

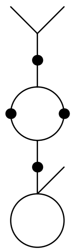
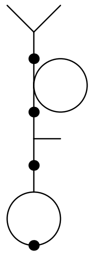
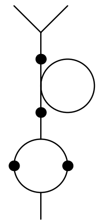
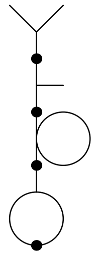
$D_3 =$ 
$E_1 =$ 
$E_2 =$ 
$F_1 =$ 

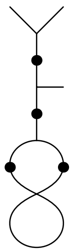
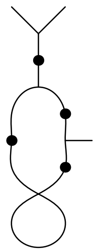
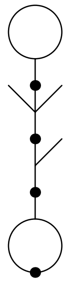
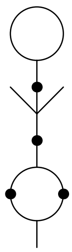
$F_2 =$ 
$F_3 =$ 
$F_4 =$ 
$G_1 =$ 

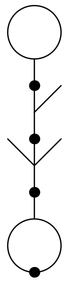
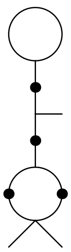
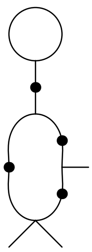
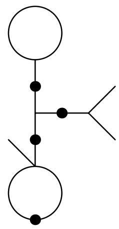
$G_2 =$ 
$H_1 =$ 
$H_2 =$ 
$H_3 =$ 

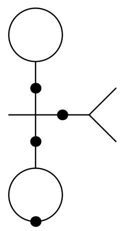
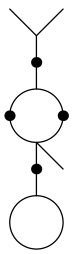
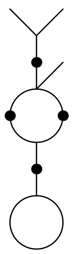
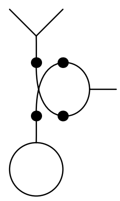
$I_1 =$ 
$I_2 =$ 
$J_1 =$ 
$J_2 =$ 

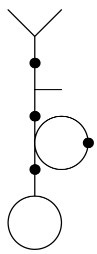
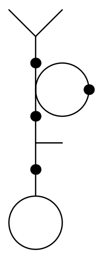
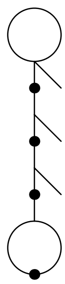
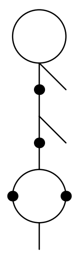
$J_3 =$ 
$K_1 =$ 
$K_2 =$ 
$L_1 =$ 

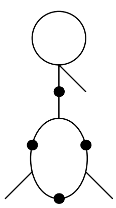
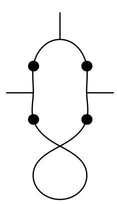
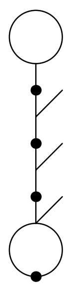
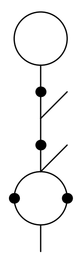
$L_2 =$  $M_1 =$  $M_2 =$  $M_3 =$ 

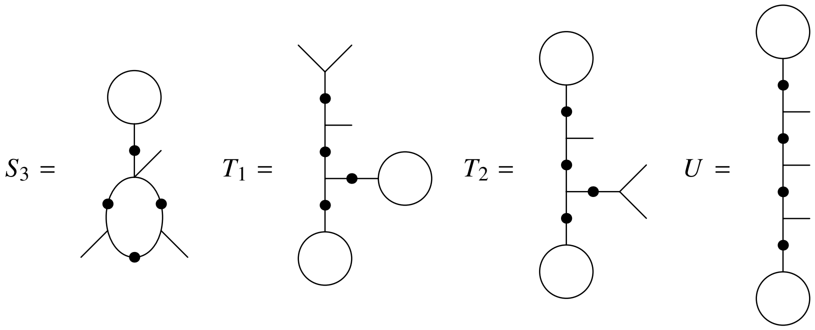
$M_4 =$  $M_5 =$  $N_1 =$  $N_2 =$ 

$N_3 =$  $N_4 =$  $N_5 =$  $O_1 =$ 

$O_2 =$  $O_3 =$  $O_4 =$  $P_1 =$ 

$P_2 =$  $P_3 =$  $Q_1 =$  $Q_2 =$ 

$Q_3 =$  $R =$  $S_1 =$  $S_2 =$ 



Appendix B. Results of calculations

	A_1	A_2	A_3	B_1	B_2	B_3	C_1	C_2	C_3	C_4
Δ_1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{12}$	0	0	0	0	0
Δ_2	0	0	0	$-\frac{1}{24}$	$-\frac{1}{36}$	0	0	0	0	0
Δ_3	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	0	0	0	0	0	0	0
Δ_4	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
Δ_5	0	$\frac{1}{24}$	0	0	0	0	0	0	0	0
Δ_6	0	$-\frac{1}{8}$	0	0	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$
Δ_7	0	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0
Δ_8	0	$\frac{1}{8}$	0	0	0	0	0	0	0	$\frac{1}{4}$
Δ_9	0	0	0	0	0	$-\frac{1}{24}$	0	0	0	0
Δ_{10}	0	$\frac{1}{8}$	0	0	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$\frac{1}{8}$
Δ_{11}	0	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	0	$-\frac{1}{8}$
Δ_{12}	0	0	0	0	0	0	0	0	0	0
Δ_{13}	0	0	0	0	0	0	0	0	0	0
Δ_{14}	0	0	0	0	0	0	0	0	0	0
Δ_{15}	0	0	0	0	0	0	0	0	0	0
Δ_{16}	$\frac{1}{12}$	0	$\frac{1}{12}$	0	0	0	0	0	0	$\frac{1}{4}$
Δ_{17}	$-\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	0	0	0	$-\frac{3}{4}$	$-\frac{3}{4}$
Δ_{18}	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0	$\frac{3}{2}$	$\frac{3}{4}$
Δ_{19}	0	0	0	0	0	$\frac{1}{12}$	0	0	0	0
Δ_{20}	0	0	0	0	0	$-\frac{1}{4}$	0	0	0	0

	D_1	D_2	D_3	E_1	E_2	F_1	F_2	F_3	F_4	G_1
Δ_1	0	0	0	0	0	0	0	0	0	0
Δ_2	0	$\frac{1}{8}$	$\frac{1}{12}$	0	$-\frac{1}{16}$	0	0	0	0	0
Δ_3	0	$-\frac{1}{4}$	$-\frac{1}{6}$	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	0
Δ_4	0	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{8}$	0	0	0	0	0	0
Δ_5	0	$-\frac{1}{8}$	0	0	$\frac{1}{16}$	0	0	$-\frac{1}{8}$	0	0
Δ_6	0	$\frac{1}{8}$	0	$\frac{1}{8}$	0	0	0	$\frac{3}{8}$	0	0
Δ_7	$\frac{1}{8}$	$\frac{1}{8}$	0	0	0	0	0	$-\frac{1}{8}$	0	0
Δ_8	0	0	0	0	0	0	0	$-\frac{1}{4}$	0	0
Δ_9	0	0	0	0	$-\frac{1}{8}$	0	0	0	0	0

	D_1	D_2	D_3	E_1	E_2	F_1	F_2	F_3	F_4	G_1
Δ_{10}	0	$-\frac{1}{8}$	0	$-\frac{1}{8}$	0	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{8}$	0	$\frac{1}{8}$
Δ_{11}	0	$\frac{1}{8}$	0	$-\frac{1}{8}$	0	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	0	0
Δ_{12}	0	0	0	0	0	0	0	0	0	0
Δ_{13}	0	0	0	0	0	0	0	0	0	0
Δ_{14}	0	0	0	0	0	0	0	0	0	0
Δ_{15}	0	0	0	0	0	0	0	0	0	0
Δ_{16}	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{8}$	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	0
Δ_{17}	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	$\frac{3}{4}$	0	$\frac{3}{4}$	$\frac{1}{4}$
Δ_{18}	0	0	$\frac{1}{4}$	0	0	0	$-\frac{3}{4}$	0	$-\frac{3}{4}$	$-\frac{1}{2}$
Δ_{19}	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0
Δ_{20}	0	0	0	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{4}$

	G_2	H_1	H_2	H_3	I_1	I_2	J_1	J_2	J_3	K_1
Δ_1	0	0	0	0	0	0	0	0	0	0
Δ_2	0	0	0	0	0	0	0	0	0	0
Δ_3	0	$-\frac{1}{8}$	$-\frac{1}{12}$	0	0	0	0	0	0	0
Δ_4	0	0	0	0	0	0	0	0	0	0
Δ_5	0	$\frac{2}{16}$	0	0	0	0	0	$\frac{1}{8}$	0	0
Δ_6	0	$-\frac{1}{8}$	0	0	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	0
Δ_7	0	$-\frac{1}{8}$	0	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0
Δ_8	0	0	0	0	$\frac{1}{8}$	0	$-\frac{1}{4}$	$\frac{1}{8}$	0	0
Δ_9	0	0	0	$\frac{1}{8}$	0	0	0	0	0	$\frac{1}{288}$
Δ_{10}	0	$\frac{1}{8}$	0	0	0	0	0	0	0	0
Δ_{11}	$\frac{1}{4}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$	0	0	0	0	0	$-\frac{1}{96}$
Δ_{12}	0	0	0	0	0	0	0	0	0	0
Δ_{13}	0	0	0	0	0	0	0	0	0	0
Δ_{14}	0	0	0	0	0	0	0	0	0	0
Δ_{15}	0	0	0	0	0	0	0	0	0	0
Δ_{16}	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{4}$	0
Δ_{17}	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
Δ_{18}	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	0	$-\frac{3}{4}$	0
Δ_{19}	0	0	0	$-\frac{1}{4}$	0	0	0	0	0	$-\frac{1}{24}$
Δ_{20}	$-\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	0	0	0	$\frac{1}{8}$

	K_2	L_1	L_2	M_1	M_2	M_3	M_4	M_5	N_1	N_2
Δ_1	0	0	0	0	0	0	0	0	0	0
Δ_2	0	0	0	0	0	$-\frac{1}{96}$	0	0	0	0
Δ_3	0	0	0	0	0	0	0	0	0	0
Δ_4	0	0	0	$-\frac{1}{96}$	$-\frac{1}{96}$	0	0	0	0	0
Δ_5	0	0	0	0	0	0	0	0	0	0
Δ_6	0	0	0	0	0	0	0	0	$\frac{1}{96}$	$\frac{1}{96}$
Δ_7	0	0	0	0	0	0	0	0	$-\frac{1}{48}$	$-\frac{1}{48}$
Δ_8	0	0	0	0	0	0	0	0	0	0
Δ_9	$\frac{1}{288}$	0	0	0	0	$\frac{1}{96}$	0	0	0	0
Δ_{10}	0	0	0	$\frac{1}{96}$	$\frac{1}{96}$	0	0	0	0	0

	K_2	L_1	L_2	M_1	M_2	M_3	M_4	M_5	N_1	N_2
Δ_{11}	$-\frac{1}{96}$	0	0	0	0	0	0	0	$-\frac{1}{96}$	$-\frac{1}{96}$
Δ_{12}	0	0	0	0	0	$-\frac{1}{8}$	0	0	0	0
Δ_{13}	0	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$
Δ_{14}	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$
Δ_{15}	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{8}$
Δ_{16}	0	0	0	0	0	$\frac{1}{8}$	$\frac{1}{96}$	$\frac{1}{48}$	0	0
Δ_{17}	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	0	0	0	0	0
Δ_{18}	0	$\frac{1}{8}$	$\frac{1}{8}$	0	0	0	0	0	0	0
Δ_{19}	$-\frac{1}{24}$	0	0	0	0	$-\frac{1}{8}$	0	0	0	0
Δ_{20}	$\frac{1}{8}$	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$

	N_3	N_4	N_5	O_1	O_2	O_3	O_4	P_1	P_2	P_3
Δ_1	0	0	0	0	0	0	0	0	0	0
Δ_2	0	0	0	0	0	0	0	0	0	0
Δ_3	0	0	0	0	0	0	0	0	0	0
Δ_4	0	0	0	0	0	0	0	0	0	0
Δ_5	$\frac{1}{96}$	0	0	$-\frac{1}{96}$	$\frac{1}{96}$	0	0	0	$-\frac{1}{96}$	$-\frac{1}{96}$
Δ_6	$-\frac{1}{48}$	0	0	$\frac{1}{96}$	$-\frac{1}{96}$	0	0	$-\frac{1}{96}$	0	0
Δ_7	$\frac{1}{96}$	0	0	0	$-\frac{1}{96}$	$-\frac{1}{96}$	0	0	0	$\frac{1}{96}$
Δ_8	0	0	0	0	0	0	0	0	0	0
Δ_9	0	0	0	0	$-\frac{1}{96}$	0	$-\frac{1}{96}$	0	$\frac{1}{96}$	0
Δ_{10}	0	0	0	0	0	0	0	0	0	0
Δ_{11}	$-\frac{1}{96}$	$-\frac{1}{96}$	$-\frac{1}{48}$	0	$\frac{1}{96}$	0	$\frac{1}{96}$	$\frac{1}{96}$	0	$\frac{1}{96}$
Δ_{12}	$-\frac{1}{8}$	0	0	$\frac{1}{8}$	$-\frac{1}{8}$	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$
Δ_{13}	$\frac{1}{4}$	0	0	$-\frac{1}{8}$	$\frac{1}{8}$	0	0	$\frac{1}{8}$	0	0
Δ_{14}	$-\frac{1}{8}$	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	0	0	0	$-\frac{1}{8}$
Δ_{15}	0	0	0	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0
Δ_{16}	0	0	0	0	0	0	0	0	0	0
Δ_{17}	0	0	0	0	0	0	0	0	0	0
Δ_{18}	0	0	0	0	0	0	0	0	0	0
Δ_{19}	0	0	0	0	$\frac{1}{8}$	0	$\frac{1}{8}$	0	$-\frac{1}{8}$	0
Δ_{20}	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	0	$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$

	Q_1	Q_2	Q_3	R	S_1	S_2	S_3	T_1	T_2	U
Δ_1	0	0	0	0	0	0	0	0	0	0
Δ_2	0	0	0	0	0	0	0	0	0	0
Δ_3	0	0	0	0	0	0	0	0	0	0
Δ_4	0	0	0	0	0	0	0	0	0	0
Δ_5	0	0	0	0	$-\frac{1}{48}$	0	0	$\frac{1}{2304}$	$\frac{1}{2304}$	$\frac{1}{1152}$
Δ_6	0	0	0	0	$\frac{1}{48}$	$-\frac{1}{48}$	0	0	0	0
Δ_7	0	0	0	0	0	$\frac{1}{48}$	$-\frac{1}{48}$	0	0	0
Δ_8	0	0	0	0	0	0	0	0	0	0
Δ_9	0	0	0	0	0	0	0	0	0	0
Δ_{10}	0	0	0	0	0	0	0	0	0	0
Δ_{11}	0	0	0	0	0	0	0	0	0	0

	Q_1	Q_2	Q_3	R	S_1	S_2	S_3	T_1	T_2	U
Δ_{12}	0	0	0	0	$\frac{1}{4}$	0	0	$-\frac{1}{96}$	$-\frac{1}{48}$	$-\frac{1}{48}$
Δ_{13}	0	0	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
Δ_{14}	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	0
Δ_{15}	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	0
Δ_{16}	0	0	0	$\frac{1}{48}$	0	0	0	0	0	0
Δ_{17}	0	0	0	0	0	0	0	0	0	0
Δ_{18}	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	0	0
Δ_{19}	0	0	0	0	0	0	0	0	0	0
Δ_{20}	0	0	0	0	0	0	0	0	0	0

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